International Journal of Theoretical & Applied Sciences, 8(2): 65-68(2016)

ISSN No. (Print): 0975-1718
ISSN No. (Online): 2249-3247

A Common Fixed Point Theorem for Separable Hilbert Space

V. K. Agrawal*, Kamal Wadhwa**, Sanjay Chaudhary** and Amit Kumar Diwakar***
[*]Govt. Home Science College, Hoshangabad, (Madhya Pradesh), INDIA.
[**]Govt. Narmada Mahavidyalaya, Hoshangabad, (Madhya Pradesh), INDIA.
[***]Research Scholar Barkatullah University, Bhopal, (Madhya Pradesh), INDIA.

(Corresponding author: Amit Kumar Diwakar)
(Received 22 August, 2016 accepted 05 October, 2016)
(Published by Research Trend, Website: www.researchtrend.net)

ABSTRACT: The object of this paper is to obtain a common fixed point theorem for four continuous random operators by considering a sequence of measurable functions satisfying certain contractive condition in separable Hilbert space.

Keywords: Hilbert Space, Common fixed point.

I. INTRODUCTION

The random fixed point theory is started by Prague school of Probability in [8]. In recent years this theory has attracted much attention of many Mathematicians some of them are [3, 4, 5, 6, and 10]. In this paper we find a new common fixed point theorem with rational inequality for two random operators defined on separable Hilbert spaces. For this we construct a sequence of measurable functions of random fixed point to the two random operators.

II.PRELIMINARY NOTES

Let H be a Hilbert space and C is a closed subset of H. Let Ω, Σ be measurable space.
Definition (a): A function $f:\Omega \times C \to C$ is Said to be random operator, if
F (. , x) : $\Omega \to C$ is measurable for all $x \epsilon C$.
Definition (b): A function $f:\Omega \times C \to C$ is Said to be measurable, if
$f'(B \cap C) \in \Sigma$ for every Borel subset B of H.
Definition (c): A function $f:\Omega \times C \to C$ is Said to be continuous, if
For fixed $t \epsilon \Omega$, f (t , .) : C $\to C$ is Continuous.
Definition (d): A measurable function $g:\Omega \to C$ is Said to be random fixed point of the random operator $f:\Omega \times C \to C$, is Continuous.
Condition 1.1. Two mappings *M,N* : $C \to C$, where C is a nonempty subset of a Hilbert space *H*, is said to satisfy Condition A if:

$$\| Mx - Ny \|^2 \leq \eta_1 [\| x - y \|^2 + \eta_2 (\| x - Mx \|^2 + \| y - Ny \|^2) + \frac{\eta_3}{2} (\| x - Ny \|^2 + \| y - Mx \|^2) + \eta_4 \left(\frac{\|x-y\|^2 + \|x-Ny\|^2 + \|x-Mx\|^2}{1 + \|x-y\| \|x-Ny\| \|x-Mx\|} \right) \quad ...(A)$$

Where

$$0 < \eta_1 + 2\eta_2 + 2\eta_3 + 2\eta_4 < 1 \quad , \eta_1, \eta_2, \eta_3, \eta_4 > 0 \quad ...(B)$$

It is well known that in a Hilbert Sapce the parallelogram law is satisfied , that is,

$$\forall x, y \in C, \quad \| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2 \quad ...(C)$$

We construct a sequence of function $\{d_n\}$ as

$$d_0 : \Omega \to C \quad ...(D)$$

is arbitrary measurable function For $\theta \in \Omega$ and n=0,1,2....,

$$d_{2n+1}(\theta) = M(\theta, d_{2n}(\theta)), \qquad d_{2n+2}(\theta) = N(\theta, d_{2n+1}(\theta)). \quad ...(E)$$

III. MAIN RESULTS

In this section, we prove a common unique fixed point theorem for two random operators in Hilbert spaces.

Theorem 2.1. Let C be a nonempty closed subset of a separable Hilbert space H. Let M and N be two continuous random operators defined on C such that for $\theta \in \Omega, M(\theta, \cdot), N(\theta, \cdot) : C \to C$ satisfy A. Then the sequence $\{d_n\}$ obtained in (D) and (E) converges to the unique common random fixed point of M and N.

Proof: For fixed $\theta \in \Omega$, n = 1,2,3...,

$$\| d_{2n+1}(\theta) - d_{2n}(\theta) \|^2 = \| M(\theta, d_{2n}(\theta)) - N(\theta, d_{2n-1}(\theta) \|^2$$
$$\leq \eta_1 \| d_{2n}(\theta) - d_{2n-1}(\theta) \|^2$$
$$+\eta_2 (\| d_{2n}(\theta) - M(\theta, d_{2n}(\theta) \|^2 + \| d_{2n-1}(\theta) - N(\theta, d_{2n-1}(\theta) \|^2)$$
$$+\frac{\eta_3}{2}(\| d_{2n}(\theta) - N(\theta, d_{2n-1}(\theta) \|^2 + \| d_{2n-1}(\theta) - M(\theta, d_{2n}(\theta) \|^2)$$
$$+\eta_4 (\frac{\|d_{2n}(\theta)-d_{2n-1}(\theta)\|^2+\|d_{2n}(\theta)-N(\theta,d_{2n-1}(\theta)\|^2+\|d_{2n}(\theta)-M(\theta,d_{2n}(\theta))\|^2}{1+\|d_{2n}(\theta)-d_{2n-1}(\theta)\|\|d_{2n}(\theta)-N(\theta,d_{2n-1}(\theta))\|\|d_{2n}(\theta)-M(\theta,d_{2n}(\theta))\|})$$
$$=\eta_1 \| d_{2n-1}(\theta) - d_{2n}(\theta) \|^2$$
$$+\eta_2 (\| d_{2n}(\theta) - d_{2n+1}(\theta) \|^2 + \| d_{2n-1}(\theta) - d_{2n}(\theta) \|^2)$$
$$+\frac{\eta_3}{2}((\| d_{2n}(\theta) - d_{2n}(\theta) \|^2 + \| d_{2n-1}(\theta) - d_{2n+1}(\theta) \|^2)$$
$$+\eta_4 (\frac{\|d_{2n}(\theta)-d_{2n-1}(\theta)\|^2+\|d_{2n}(\theta)-d_{2n}(\theta)\|^2+\|d_{2n}(\theta)-d_{2n+1}(\theta)\|^2}{1+\|d_{2n}(\theta)-d_{2n-1}(\theta)\|\|d_{2n}(\theta)-d_{2n}(\theta)\|\|d_{2n}(\theta)-d_{2n+1}(\theta)\|})$$
$$\leq \eta_1 \| d_{2n}(\theta) - d_{2n-1}(\theta) \|^2$$
$$+\eta_2 (\| d_{2n}(\theta) - d_{2n+1}(\theta) \|^2 + \| d_{2n-1}(\theta) - d_{2n}(\theta) \|^2)$$
$$+\frac{\eta_3}{2}(\| d_{2n-1}(\theta) - d_{2n+1}(\theta) \|^2)$$
$$+\eta_4 (\| d_{2n}(\theta) - d_{2n-1}(\theta) \|^2 + \| d_{2n}(\theta) - d_{2n+1}(\theta) \|^2$$
$$=\eta_1 \| d_{2n}(\theta) - d_{2n-1}(\theta) \|^2$$
$$+\eta_2 (\| d_{2n}(\theta) - d_{2n+1}(\theta) \|^2 + \| d_{2n-1}(\theta) - d_{2n}(\theta) \|^2)$$
$$+\frac{\eta_3}{2}(\| d_{2n-1}(\theta) - d_{2n}(\theta) + d_{2n}(\theta) - d_{2n+1}(\theta) \|^2)$$
$$+\eta_4 (\| d_{2n}(\theta) - d_{2n-1}(\theta) \|^2 + \| d_{2n}(\theta) - d_{2n+1}(\theta) \|^2$$
$$\leq \eta_1 \| d_{2n}(\theta) - d_{2n-1}(\theta) \|^2$$
$$+\eta_2 (\| d_{2n}(\theta) - d_{2n+1}(\theta) \|^2 + \| d_{2n-1}(\theta) - d_{2n}(\theta) \|^2)$$
$$+\eta_3 (\| d_{2n-1}(\theta) - d_{2n}(\theta) \|^2 + \| d_{2n}(\theta) - d_{2n+1}(\theta) \|^2)$$
$$-\frac{\eta_3}{2} \| (d_{2n-1}(\theta) - d_{2n}(\theta)) - (d_{2n}(\theta) - d_{2n+1}(\theta)) \|^2$$
$$+\eta_4 (\| d_{2n}(\theta) - d_{2n-1}(\theta) \|^2 + \| d_{2n}(\theta) - d_{2n+1}(\theta) \|^2$$

(by parallelogram law C)

$$\leq (\eta_1 + \eta_2 + \eta_3 + \eta_4) \| d_{2n}(\theta) - d_{2n-1}(\theta) \|^2$$
$$+(\eta_2 + \eta_3 + \eta_4) \| d_{2n}(\theta) - d_{2n+1}(\theta) \|^2 \quad ..(A.1)$$

Therefore:

$$\| d_{2n+1}(\theta) - d_{2n}(\theta) \|^2 \leq \frac{(\eta_1+\eta_2+\eta_3+\eta_4)}{1-\eta_2-\eta_3-\eta_4} \| d_{2n}(\theta) - d_{2n-1}(\theta) \|^2. \quad ..(A.2)$$

For $\theta \in \Omega$, $n = 1,2,3 \dots$,

$$\| d_{2n}(\theta) - d_{2n-1}(\theta) \|^2 = \| M(\theta, d_{2n-1}(\theta)) - N(\theta, d_{2n-2}(\theta)) \|^2$$
$$\leq \eta_1 \| d_{2n-1}(\theta) - d_{2n-2}(\theta) \|^2$$
$$+\eta_2 (\| d_{2n-1}(\theta) - M(\theta, d_{2n-1}(\theta)) \|^2 + \| d_{2n-2}(\theta) - N(\theta, d_{2n-2}(\theta)) \|^2)$$
$$+\frac{\eta_3}{2}(\| d_{2n-1}(\theta) - N(\theta, d_{2n-2}(\theta)) \|^2 + \| d_{2n-2}(\theta) - M(\theta, d_{2n-1}(\theta)) \|^2)$$
$$+\eta_4 (\frac{\|d_{2n-1}(\theta)-d_{2n-2}(\theta)\|^2+\|d_{2n-1}(\theta)-N(\theta,d_{2n-2}(\theta))\|^2+\|d_{2n-1}(\theta)-M(\theta,d_{2n-1}(\theta))\|^2}{1+\|d_{2n-1}(\theta)-d_{2n-2}(\theta)\|\|d_{2n-1}(\theta)-N(\theta,d_{2n-2}(\theta))\|\|d_{2n-1}(\theta)-M(\theta,d_{2n-1}(\theta)\|})$$

(By condition A)

$$=\eta_1 \| d_{2n-1}(\theta) - d_{2n-2}(\theta) \|^2$$
$$+\eta_2 (\| d_{2n-1}(\theta) - d_{2n}(\theta) \|^2 + \| d_{2n-2}(\theta) - d_{2n-1}(\theta) \|^2)$$
$$+\frac{\eta_3}{2}(\| d_{2n-2}(\theta) - d_{2n}(\theta) \|^2)$$

$+\eta_4(\frac{\| d_{2n-1}(\theta)-d_{2n-2}(\theta)\|^2+\| d_{2n-1}(\theta)-d_{2n}(\theta)\|^2}{1})$

$\leq \eta_1 \ \| d_{2n-1}(\theta)-d_{2n-2}(\theta) \|^2$

$+\eta_2 \ (\| d_{2n-1}(\theta)-d_{2n}(\theta) \|^2 + \| d_{2n-2}(\theta)-d_{2n-1}(\theta) \|^2)$

$+\eta_3(\| d_{2n-2}(\theta)-d_{2n-1}(\theta) \|^2 + \| d_{2n-1}(\theta)-d_{2n}(\theta) \|^2)$

$-\frac{\eta_3}{2}(\| d_{2n-2}(\theta)-d_{2n-1}(\theta)) - \| (d_{2n-1}(\theta)-d_{2n}(\theta) \|^2)$

$+\eta_4(\| d_{2n-1}(\theta)-d_{2n-2}(\theta) \|^2 + \| d_{2n-1}(\theta)-d_{2n}(\theta) \|^2)$

(by parallelogram law)

$\leq (\eta_1+\eta_2+\eta_3+\eta_4) \| d_{2n-1}(\theta)-d_{2n-2}(\theta) \|^2$

$+(\eta_2+\eta_3+\eta_4) \ \| d_{2n-1}(\theta)-d_{2n}(\theta) \|^2.$ ..(A.3)

Therefore:

$\| d_{2n}(\theta)-d_{2n-1}(\theta) \|^2 \leq (\frac{\eta_1+\eta_2+\eta_3+\eta_4}{1-\eta_2-\eta_3-\eta_4}) \| d_{2n-1}(\theta)-d_{2n-2}(\theta) \|^2$ ...(A.4)

Equation (A.2) and (A.4) jointly imply that for all $\theta \in \Omega$ and n=0,1,2,3...,

$\| d_n(\theta)-d_{n+1}(\theta) \|^2 \leq (\frac{\eta_1+\eta_2+\eta_3+\eta_4}{1-\eta_2-\eta_3-\eta_4}) \| d_{n-1}(\theta)-d_n(\theta) \|^2$ ...(A.5)

Again from (B) it follows that:

$0 < (\frac{\eta_1+\eta_2+\eta_3+\eta_4}{1-\eta_2-\eta_3-\eta_4}) < 1.$ ...(A.6)

From (A.5) and (A.6) it follows that for $\in \Omega$, $\{d_n(\theta)\}$ is a Cauchy sequence and hence is convergent in the Hilbert space H.

For $\theta \in \Omega$, let

$\{d_n(\theta)\} \to d(\theta)$ as $n \to \infty$ ...(A.7)

Since C is closed , d is a function from C to C.

For $\in \Omega$,

$\| d(\theta)-N(\theta,d(\theta) \|^2 = \| d(\theta)-d_{2n}(\theta)+(d_{2n}(\theta)-M(\theta,d(\theta) \|^2$

$\leq 2 \ \| d(\theta)-d_{2n}(\theta) \|^2 + 2 \ \| d_{2n}(\theta)-M(\theta,g(\theta)) \|^2,$

(by parallelogram law)

$=2\| d(\theta)-d_{2n}(\theta) \|^2 + 2 \ \| N(\theta, \ d_{2n-1}(\theta)-M(\theta,d(\theta)) \|^2$

$=2 \ \| d(\theta)-d_{2n}(\theta) \|^2 + 2[\eta_1 \ \| d_{2n-1}(\theta)-d(\theta) \|^2 +$

$2\eta_2(\| d_{2n-1}(\theta)-N(\theta,d_{2n-1}(\theta)) \|^2 + \| d(\theta)-M(\theta,d(\theta) \|^2)$

$+\eta_3(\| d_{2n-1}(\theta)-M(\theta,d(\theta)) \|^2 + \| d(\theta)-N(\theta,d_{2n-1}(\theta)) \|^2)$

$+\eta_4(\frac{\| d_{2n}(\theta)-d_{2n-1}(\theta)\|^2+\| d_{2n}(\theta)-d_{2n}(\theta)\|^2+\| d_{2n}(\theta)-d_{2n+1}(\theta)\|^2}{1+\| d_{2n}(\theta)-d_{2n-1}(\theta)\|\| d_{2n}(\theta)-d_{2n}(\theta)\|\| d_{2n}(\theta)-d_{2n+1}(\theta)\|})]$

$=2 \ \| d(\theta)-d_{2n}(\theta) \|^2 + 2[\eta_1 \ \| d_{2n-1}(\theta)-d(\theta) \|^2$

$+2\eta_2\big(\| d_{2n-1}(\theta)-d_{2n}(\theta) \|^2 + \| d(\theta)-M\big(\theta,d(\theta)\big) \|^2\big)$

$+\eta_3(\| d_{2n-1}(\theta)-M(\theta,d(\theta) \|^2 + \| d(\theta)-d_{2n}(\theta) \|^2)$

$+\eta_4(\frac{\| d_{2n-1}(\theta)-d_{2n-2}(\theta)\|^2+\| d_{2n-1}(\theta)-d_{2n}(\theta)\|^2}{1})]$

Making $\to \infty$, $d_n\theta \to d(\theta)$

$\| d(\theta)-N(\theta,d(\theta) \|^2 \leq (2\eta_2+\eta_3+2\eta_4) \ \| d(\theta)-N(\theta,d(\theta) \|^2$ ...(A.9)

Since $0 < 2\eta_2+\eta_3+2\eta_4 < 1$ (by B), we have for all $\theta \in \Omega$,

$N(\theta,d(\theta)) = d(\theta)$ . ...(A.10)

In an exactly similar way we can prove that for all $\theta \in \Omega$,

$M(\theta,d(\theta)) = d(\theta)$ ...(A.11)

Again, if $A : \Omega \times C \to C$ is a continuous random operator on a nonempty subset C of a separable Hilbert space H, then for any measurable function $f : \Omega \to C$,

the function $h(\theta) = A(\theta, f(\theta))$ is also measurable .

Therefore the sequence of measurable function $\{d_n\}$ converges to measurable function with:

M$(\theta,h(\theta))$= $d(\theta)$), N$(\theta,d(\theta))$= $d(\theta)$) ....(A.12)

This Completes the proof of the theorem.

REFERENCES

[1]. Beg, I. and Shahzad, N. (1994). Random approximations and random fixed point theorems, *J. Appl. Math. Stochastic Anal.,* **7**, No. 2, 145-150.
[2]. Bharucha-Reid. A. T., (1996). Fixed point theorems in probabilistic analysis, *Bull. Amer. Math. Soc*., **82**, 611-645.
[3]. Bharucha-Reid. A. T., (1972). Random Integral equations Academic Press, New York,.
[4]. Choudhury, B.S. (1995). Convergence of a random iteration scheme to a random fixed point. *J. Appl. Math. Stochastic Anal.,* **8**, No. 2, 139-142.
[5]. Choudhury, B.S. and Ray, M. (1999). Convergence of an iteration leading to a solution of a random operator equation, *J. Appl. Math. Stochastic Anal.,* **12**, No. 2, 161-168.
[6]. Choudhury, B.S and Upadhyay, A. (1999). An iteration leading to a random solutions and fixed points of operators, *Soochow J. Math*., **25**, No.4, 395-400.
[7]. Choudhury, B.S. (2002). A common unique fixed point theorem for two random operators in Hilbert spaces, *IJMMS*, **32**: 3, 177- 182
[8]. Hans. P, (1957). Random fixed point theorems, Transactions of the first Prague Conference on Information Theory, Statistical Decision Functions, Random Process, pp. 105-125.
[9]. Pagey .S. S and Malviya. Neeraj, (2010). A Common Unique Random Fixed Point Theorem for rational inequality in Hilbert Space, *Int. Journal of Math. Analysis,* Vol. **4**, no. 3, 133 – 141(2010).
[10]. Regan, D.O, (1998). "Fixed points and random fixed points for weakly inward approximable maps", *Proceedings of the American Mathematical Society,* **126**(10), (1998), 3045-3053.